MCV4U Lesson 6

Operations with Algebraic Vectors

Most of the operations for algebraic vectors below are defined for vectors in R^3. It should be simple to apply these definitions to vectors in R^2 as well.

Vector Equality - Let $\vec{u} = (a_1, b_1, c_1)$ and $\vec{v} = (a_2, b_2, c_2)$. We say that $\vec{u} = \vec{v}$ if and only if

$a_1 = a_2$

$b_1 = b_2$

$c_1 = c_2$

Scalar Multiplication

Let $\vec{v} = (a, b, c)$ be any vector in R^3. Given scalar $k \epsilon R$ then:

$k\vec{v} = (ka, kb, kc)$

Vector Addition/Subtraction

How do we add two algebraic vectors?

Let $\vec{u} = (a_1, b_1, c_1)$ and $\vec{v} = (a_2, b_2, c_2)$

$\vec{u} + \vec{v} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

Example: Given $\vec{u} = (3,1,0)$ and $\vec{v} = (1,-1,3)$ Find $3\vec{u} + 4\vec{v}$:

$= 3(3,1,0) + 4(1,-1,3)$

$= (9,3,0) + (4,-4,12)$

$= (13,-1,12)$

Vector Joining 2 Points

Often we want to define an algebraic vector that goes through to points.

In R^2: The point-to-point vector $\overrightarrow{PQ}$ for points P(a_1, b_1) and Q(a_2,b_2) is given by:

$$\overrightarrow{PQ} = (a_2 - a_1, b_2 - b_1)$$

In R^3: The point-to-point vector $\overrightarrow{PQ}$ for points P(a_1, b_1, c_1) with Q(a_2,b_2, c_2) is given by:

$$\overrightarrow{PQ} = (a_2 - a_1, b_2 - b_1, c_2 - c_1)$$

Collinear

~~Parallel~~ Vectors

Two vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if:

$\vec{u} = k\vec{v}$ for some $k \in R$.

Example: Determine whether $\overrightarrow{AB} \| \overrightarrow{CD}$ given A(2,0), B(3,6), C(3, 1) and D(5,-5).

$\overrightarrow{AB} = (3-2, 6-0)$ $\qquad$ $\overrightarrow{CD} = (2,-6)$

$\overrightarrow{AB} = (1, 6)$

Find k such that $\overrightarrow{AB} = k\overrightarrow{CD}$?

We call parallel vectors "collinear". Collinear refers to the fact no ~~that~~ they can be drawn on the same line.

Collinear

Three points, A, B and C are collinear if and only if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in R$.

Example: Determine whether or not the points A(9, -6, 15), B(-3, 2, -5) and C(6, -4, 10) are collinear.

$\overrightarrow{AB} = (-12, 8, -20)$ $\qquad$ $\overrightarrow{BC} = (9, -6, 15)$

$\overrightarrow{AB} = k\overrightarrow{CD}$ for some $k \in R$?

$(-12, 8, -20) = k(9, -6, 15)$

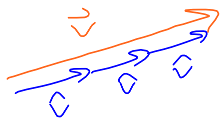
$k = -\frac{3}{4}$ $\qquad$ yes, A, B and C are collinear

Unit Vectors

A unit vector has a magnitude of one unit. It is denoted by the ^ symbol.

$\hat{v} \leftarrow$ unit vector in same dir. as $\vec{v}$.

Any vector $\vec{v}$, can be expressed as a scalar product of its magnitude and a unit vector having the same direction.



$|\vec{v}| = 3$

$\vec{v} = 3\hat{v}$

$\vec{v} = |\vec{v}|\,\hat{v}$

Example: given $\vec{v} = (4,3)$:

$|\vec{v}| = \sqrt{4^2+3^2} = 5$

$\vec{v} = 5\hat{v}$

We can rearrange this definition of a vector to derive a **definition of a unit vector.**

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

So we can use the above process to make a vector into a unit vector having the same direction. We call this **normalizing a vector**.

Example: Normalize the vector $\vec{v} = (2,-3,-6)$ Find $\hat{v}$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

$$\hat{v} = \frac{(2,-3,-6)}{\sqrt{2^2+(-3)^2+(-6)^2}}$$

$$\hat{v} = \frac{(2,-3,-6)}{7}$$

$$\hat{v} = \left(\tfrac{2}{7}, \tfrac{-3}{7}, -\tfrac{6}{7}\right)$$

So if we travel one unit from the origin along the vector $\vec{v}$, we will be at point:

$$\left(\tfrac{2}{7}, \tfrac{-3}{7}, -\tfrac{6}{7}\right)$$

Note that the components of a unit vector $\hat{v}$, are also the direction cosines for any vector that is a multiple of v! This tells us that:

$\vec{v} = (a,b,c)$

$|\hat{v}| = 1$ and

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

$$\hat{v} = \frac{(a,b,c)}{|\vec{v}|}$$

$$\hat{v} = \left(\frac{a}{|\vec{v}|}, \frac{b}{|\vec{v}|}, \frac{c}{|\vec{v}|}\right)$$

Use algebraic vectors (i.e. components) to prove the distributive law for vector addition/scalar multiplication. i.e. prove that $k(\vec{a}+\vec{b}) = k\vec{a} + k\vec{b}$

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

$$k(\vec{a}+\vec{b})$$
$$= k((a_1, a_2, a_3) + (b_1, b_2, b_3))$$
$$= k(a_1+b_1, a_2+b_2, a_3+b_3)$$
$$= (k(a_1+b_1), k(a_2+b_2), k(a_3+b_3))$$
$$= (ka_1+kb_1, ka_2+kb_2, ka_3+kb_3)$$

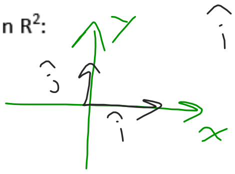
$$k\vec{a} + k\vec{b}$$
$$= k(a_1, a_2, a_3) + k(b_1, b_2, b_3)$$
$$= (ka_1, ka_2, ka_3) + (kb_1, kb_2, kb_3)$$
$$= (ka_1+kb_1, ka_2+kb_2, ka_3+kb_3)$$

$$LS = RS$$

Unit Vector Notation

We define 3 standard unit vectors in the direction of x, y and z axes.

In R^2:



$\hat{i} = (1, 0)$ $\hat{j} = (0, 1)$

In R^3:

$\hat{i} = (1, 0, 0)$

$\hat{j} = (0, 1, 0)$

$\hat{k} = (0, 0, 1)$

Any algebraic vector can be defined in either component form or in unit vector notation.

Examples

(4, -8)

$= 4\hat{i} - 8\hat{j}$

(1, -2, 7)

$= \hat{i} - 2\hat{j} + 7\hat{k}$

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