MCV4U

The Cross Product

The cross product is an operation that given two vectors $\vec{a}$ and $\vec{b}$ will return a vector that is perpendicular to both $\vec{a}$ and $\vec{b}$. Hence, the cross product is only defined in 3-space. WHY?

The cross product or vector product of $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is given by:

$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & b_1 & b_2 & b_3 \end{array}$$

Example: Let $\vec{u} = (1,2,-4)$ and $\vec{v} = (2,-3,1)$. Find $\vec{u} \times \vec{v}$. Verify this new vector is perpendicular to $\vec{u}$ and $\vec{v}$.

$\vec{u} \times \vec{v} = (-10, -9, -7)$

$$\begin{array}{cccccc} 1 & 2 & -4 & 1 & 2 & -4 \\ 2 & -3 & 1 & 2 & -3 & 1 \end{array}$$

$\vec{u} \cdot (-10, -9, -7)$

$= 0$ ✓

$\vec{v} \cdot (-10, -9, -7)$

$= 0$

If $\vec{a}$ and $\vec{b}$ are non-collinear vectors in R^3 then every vector that is perpendicular to both $\vec{a}$ and $\vec{b}$ will have the form $k(\vec{a} \times \vec{b})$ for some $k \epsilon R$.

We won't prove the above statement. But we can demonstrate that the cross product does in fact produce a vector perpendicular to the two given vectors. Let $\vec{v} = \vec{a} \times \vec{b}$. Verify that $\vec{v}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

$\vec{v} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$

$\vec{v} \cdot \vec{a} = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)$

$= \cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_1a_2b_3} + \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1}$

$= 0$

$\therefore \vec{v} \perp \vec{a}$

Example: Find 3 vectors perpendicular to both $\vec{x} = (8,4,0)$ and $\vec{y} = (-2,1,6)$

$\vec{x} \times \vec{y} = (24, -48, 16)$

$(240, -480, 160)$

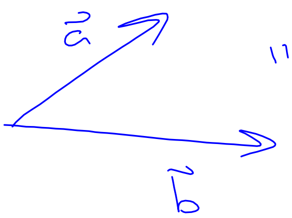
$(12, -24, 8)$

$(-3, 6, -2)$ $(3, -6, 2)$

How can we express the cross-product geometrically?

$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$

direction follows the "right hand rule"



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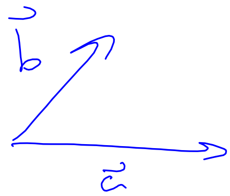

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Direction of cross product is given by a "right-handed system". Wrap fingers from a to b, thumb points in direction of cross product (see page 183-184)

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