Assignment 1 - Solutions

1. a) $\vec{AB}+\vec{AC}+\vec{AD}+\vec{AE}+\vec{AF}$

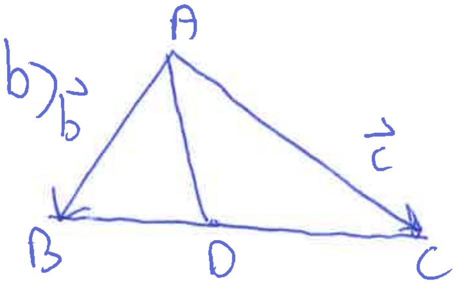
$= \vec{ED}+\vec{AC}+\vec{AD}+\vec{AE}+\vec{CD}$ since $\vec{AB}=\vec{ED}$, $\vec{AF}=\vec{CD}$

$= \vec{AC}+\vec{AD}+\vec{AE}+\vec{ED}+\vec{CD}$

$= \vec{AC}+\vec{CD}+\vec{AE}+\vec{ED}+\vec{AD}$

$= \vec{AD}+\vec{AD}+\vec{AD}$ ($\triangle$ law x2)

$= 3\vec{AD}$

b) 

$\vec{AD} = \vec{AB}+\vec{BD}$

$= \vec{AB}+\frac{1}{2}\vec{BC}$

$= \vec{AB}+\frac{1}{2}(\vec{BA}+\vec{AC})$

$= \vec{AB}+\frac{1}{2}\vec{BA}+\frac{1}{2}\vec{AC}$

$= \vec{AB}-\frac{1}{2}\vec{AB}+\frac{1}{2}\vec{AC}$

$= \frac{1}{2}\vec{AB}+\frac{1}{2}\vec{AC}$

$= \frac{1}{2}\vec{b}+\frac{1}{2}\vec{c}$

$\square$

2.a) $LS = \vec{x} \cdot \vec{y}$ $\quad$ RCS ~~$\frac{1}{2}$~~

$= |\vec{x}||\vec{y}| \cos\theta$

$RS = \frac{1}{2}\left(|\vec{x}+\vec{y}|^2 - |\vec{x}|^2 - |\vec{y}|^2\right)$

$= \frac{1}{2}\left(|\vec{x}|^2 + |\vec{y}|^2 + 2|\vec{x}||\vec{y}|\cos\theta - |\vec{x}|^2 - |\vec{y}|^2\right)$

$= \frac{1}{2}\left(2|\vec{x}||\vec{y}|\cos\theta\right)$

$= |\vec{x}||\vec{y}|\cos\theta$

$= LS$ $\quad \square$.

b) Let $\vec{x} = (x_1, x_2, x_3)$ $\vec{y} = (y_1, y_2, y_3)$

~~RS=~~

$RS = \frac{1}{2}\left(|(x_1+y_1, x_2+y_2, x_3+y_3)|^2 - |(x_1, x_2, x_3)|^2 - |(y_1, y_2, y_3)|^2\right)$

$= \frac{1}{2}\left((x_1+y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2 - (x_1^2+x_2^2+x_3^2) - (y_1^2+y_2^2+y_3^2)\right)$

$= \frac{1}{2}\left(x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2 + x_3^2 + 2x_3y_3 + y_3^2 - x_1^2 - x_2^2 - x_3^2 - y_1^2 - y_2^2 - y_3^2\right)$

$= \frac{1}{2}\left(2x_1y_1 + 2x_2y_2 + 2x_3y_3\right)$

$= x_1y_1 + x_2y_2 + x_3y_3$

$= \vec{x} \cdot \vec{y}$

3.

a) $\vec{a} = (2,3,7) \quad \vec{b} = (-4,y,-14)$

perpendicular $\quad \vec{a} \cdot \vec{b} = 0$

$$(2,3,7)\cdot(-4,y,-14) = 0$$

$$-8 + 3y - 98 = 0$$

$$3y = 106$$

$$y = \frac{106}{3}$$

b) collinear if $k(2,3,7) = (-4,y,-14)$ for some $k \in \mathbb{R}$.

$$2k = -4 \qquad 7k = -14$$

$$k = -2 \qquad k = -2 \checkmark$$

$$\therefore\ 3(-2) = y$$

$$y = -6.$$

b)

let $\vec{v} = \vec{x} \times \vec{y}$

$$= (2,0,-1) \times (-3,-4,1)$$

$$= (-4,1,-8)$$

$$\begin{array}{cccccc} 2 & 0 & -1 & 2 & 0 & -1 \\ -3 & -4 & 1 & -3 & -4 & 4 \end{array}$$

Find $\hat{v}$

$$\hat{v} = \frac{(-4,1,-8)}{\sqrt{(-4)^2 + 1^2 + (-8)^2}}$$

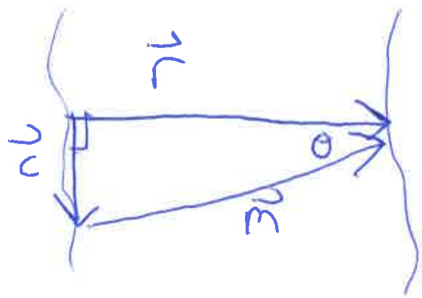
$$\hat{v} = \frac{(-4,1,-8)}{\sqrt{81}}$$

$$\hat{v} = \left(\frac{-4}{9}, \frac{1}{9}, \frac{-8}{9}\right)$$

$\therefore$ 2 unit vectors are $(-4/9, 1/9, -8/9)$ and $(4/9, -1/9, 8/9)$

there are no more.

4.



Let $\vec{c}$ = current
$\vec{m}$ = water velocity.
$\vec{r}$ = resultant

So that

$$\vec{r} = \vec{c} + \vec{m}$$

$$\sin\theta = \frac{|\vec{c}|}{|\vec{m}|}$$

$$\sin\theta = \frac{6}{20}$$

$$\theta \doteq 17.5^\circ$$

$$|\vec{m}|^2 = |\vec{c}|^2 + |\vec{r}|^2$$

$$20^2 = 6^2 + |\vec{r}|^2$$

$$|\vec{r}| = \sqrt{364}$$

$$|\vec{r}| \doteq 19.1 \text{ km/h.}$$

$\therefore$ She needs to head 17.5° upstream

$s = \frac{d}{t}$
$st = d$
$t = \frac{d}{s}$

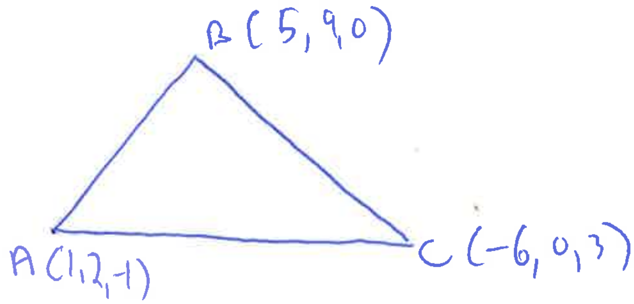
It will take her $\dfrac{2 \text{ km}}{19.1 \text{ km/h}}$

$= 0.10471$ hours

or 6.28 minutes

5.

a)



$\vec{AB} = (4,7,1)$

$\vec{AC} = (-7,-2,4)$

$$\text{Area} = \frac{|\vec{AB} \times \vec{AC}|}{2}$$

$$= \frac{|(4,7,1)\times(-7,-2,4)|}{2}$$

$$\begin{array}{cccccc} 4 & 7 & 1 & 4 & 7 & 1 \\ -7 & -2 & 4 & -7 & -2 & 4 \end{array}$$

$-8-(-49)$

$-8+49$

$$= \frac{|(30,-23,41)|}{2}$$

$$= \frac{\sqrt{30^2+(-23)^2+41^2}}{2}$$

$$= \frac{\sqrt{3110}}{2} \text{ sq. units.}$$

b) $(\hat{a}-5\hat{b})\cdot(\hat{a}-\hat{b})=0$

$|\hat{a}|^2 - 6\,\hat{a}\cdot\hat{b} + 5|\hat{b}|^2 = 0$

$(1) - 6(1)(1)\cos\theta + 5(1) = 0$

$-6\cos\theta + 6 = 0$

$\cos\theta - 1 = 0$

$\cos\theta = 1$

$\therefore \theta = 0°$

6. a) $V = |\vec{AB} \cdot (\vec{AD} \times \vec{AC})|$ $\qquad \vec{AB} = (4,2,-4)$

$\vec{AC} = (-1,10,-2)$

$V = |(4,2,-4) \cdot ((1,6,4) \times (-1,10,-2))|$ $\qquad \vec{AD} = (1,6,4)$

$V = |(4,2,-4) \cdot (-52,-2,16)|$

$V = 276$ cu. units.

b) $\vec{AE} = \vec{AB} + \vec{AC}$ // gram law.

$\vec{AE} = (4,2,-4) + (-1,10,-2)$

$\vec{AE} = (3,12,-6)$

$E(a,b,c)$ $\qquad \vec{AE} = (a-1, b+3, c-2)$

$\therefore a = 4, \ b = 9, \ c = -4$

$E(4,9,-4)$

c) $\cos\theta = \dfrac{\vec{AE} \cdot \vec{BC}}{|\vec{AE}|\,|\vec{BC}|}$ $\qquad \vec{BC} = (-5,8,2)$

$\cos\theta = \dfrac{(3,12,-6) \cdot (-5,8,2)}{\sqrt{3^2+12^2+6^2}\,\sqrt{5^2+8^2+2^2}}$

$\cos\theta = \dfrac{69}{\sqrt{189}\,\sqrt{93}}$ $\qquad \theta \doteq 58.6^\circ$

7. $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$

a) $= \vec{a}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \vec{b}\times\vec{b}$

$= \vec{0} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - 0$

$= \vec{a}\times\vec{b} - \vec{b}\times\vec{a}$

$= \vec{a}\times\vec{b} - (-\vec{a}\times\vec{b})$

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{b}$

$= 2\vec{a}\times\vec{b}$ (or use components)

b) $\vec{c} = \vec{a}\times\vec{b}$ $\therefore$ $\vec{c}$ is $\perp$ to $\vec{a}$ and $\vec{b}$ and any vector in the same plane as $\vec{a}$ and $\vec{b}$.

$\vec{d}$ is in the same plane as $\vec{a}$ and $\vec{b}$.

$\therefore$ $\vec{c}\cdot\vec{d} = 0$ since $\vec{c} \perp \vec{d}$.