The Second Derivative

We will start with a quick sketch of $f(x) = x^3 - 6x^2 + 3$ by determining end behavior and turning points.

$f'(x) = 3x^2 - 12x$ $\qquad$ $x = 0$, $x = 4$

$0 = 3x^2 - 12x$ $\qquad$ $f(0) = 3$ $\quad$ $f(4) = -29$

$0 = 3x(x - 4)$ $\qquad$ $(0, 3)$ $\quad$ $(4, -29)$

$f''(x) = 6x - 12$ $\qquad$ $x = 2$

$0 = 6x - 12$ $\qquad$ $f''(0) \neq 0$

$6x = 12$ $\qquad$ $f''(4) \neq 0$

$f'(x)$

$f''(x)$

The Second Derivative

The second derivative of *f(x)* is defined as the function $\boxed{f''(x) = \frac{d}{dx}(f'(x))}$

In Leibniz notation this can be written as $\boxed{\frac{d^2y}{dx^2}}$

"The derivative of the derivative" or "the rate of change of the slope of the tangent line to f(x)."

$\frac{dy}{dx}$

Concavity

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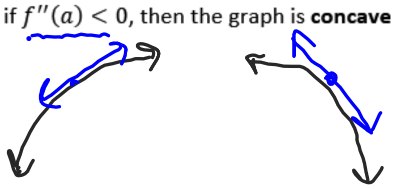
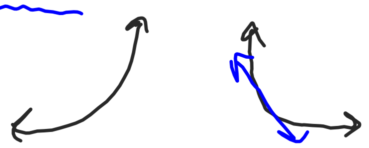
The second derivative tells us about the concavity of a function's graph.

If if $f''(a) < 0$, then the graph is **concave down** at x = a.



tangents above curve

If $f''(a) > 0$, then the graph is **concave up** at x = a.



tangents below curve

We have already learned that f'(x) = 0 is a possible turning point (local max or min). This is possible because f'(x) may change sign from positive to negative or vice versa.

A point of inflection is a point where the graph changes from concave down to concave up or vice versa. In order for this to happen f''(x) must change from positive to negative or vice versa.

If f(x) has a point of inflection at x = c then f''(c) = 0.

Find the point(s) of inflection on the graph from the first page.

Second Derivative Test

If $f'(c) = 0$ and $f''(c) \neq 0$ then there is a local max or min at x = c.

Combine this with the information above and we can say that:

If $\underline{f'(c) = 0}$ and $\underline{f''(c) > 0}$, then $f(x)$ has a local minimum at x = c.

If $f'(c) = 0$ and $\underline{f''(c) < 0}$, then $f(x)$ has a local maximum at x = c.

Example – Use the second derivative test to determine the turning points of $f(x) = x^3 - 3x^2 - 9x + 27$.

$f'(x) = 3x^2 - 6x - 9$

$0 = 3(x^2 - 2x - 3)$

$0 = 3(x-3)(x+1)$

C.P. $x = 3$, $x = -1$

$f''(x) = 6x - 6$

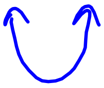
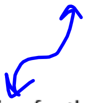
$f''(3) = 6(3) - 6$
$= 12$

$f''(-1) = 6(-1) - 6$
$= -12$

$f''(3) > \underline{0}$
local min

$f''(-1) < 0$
local max.

You do need to be careful with the second derivative test. Although a point of inflection at x = c means $f''(c) = 0$, the opposite is not true. That is to say if $f''(c) = 0$, it does not necessarily mean there is a point of inflection at x = c. In fact it is very common to have $f'(c) = f''(c) = 0$, and lots of possibilities in this situation:

$y = x^4$   $y = x^3$ $\frac{dy}{dx} = 3x^2$

Try to find turning points and points of inflection for the graph of $f(x) = x^3 - 6x^2 + 12x - 18$. Notice the second derivative test fails to assist us this example.

$f'(x) = 3x^2 - 12x + 12$

C.P. $0 = 3x^2 - 12x + 12$

$0 = x^2 - 4x + 4$

$0 = (x-2)^2$

$x = 2$

$f''(x) = 6x - 12$

$0 = 6x - 12$

$6x = 12$

$x = 2$

check $f'(1) = 3 > 0$

$f'(2) = 0$

$f'(3) = 3 > 0$

not a T.P.

point of inflection