An Algorithm for Curve Sketching

Lesson 3

Based on what we have learned so far (including material in Advanced Functions last semester), we can follow the following steps to sketch the graph of any function.

1) Determine any intercepts (provided they are relatively easy to calculate).
2) Determine the end behavior of the function (may provide you with a horizontal asymptote)
3) Determine if any vertical asymptotes exist (if the function is a rational function)
4) Use the first derivative to find all critical points.
5) Find the second derivative, and any locations where it is zero.
6) Use the information from steps 4) and 5) to determine location of all turning points (local maximum and minimum values) and points of inflection.
7) Sketch the function.

Examples

1) $f(x) = x^3 - x^2 - x + 1$

2) $f(x) = \frac{x}{x^2+1}$

3) $f(x) = \frac{x^3}{x^3+1}$

Assigned Problems

Sketch each of the following.

1) $f(x) = \frac{1}{x^3+1}$

2) $f(x) = \frac{x^2}{x^2+1}$

3) $f(x) = \frac{x^2-1}{x^4}$

4) $f(x) = \frac{x}{x^2-1}$

Example 1

$f(x) = x^3 - x^2 - x + 1$

① intercepts $f(0) = 1$ $(0,1)$

$$0 = x^3 - x^2 - x + 1$$
$$0 = x^2(x-1) - 1(x-1)$$
$$0 = (x-1)(x^2-1)$$
$$0 = (x-1)(x-1)(x+1)$$
$$x = 1 \text{ and } x = -1$$

intercepts: $(1,0)$, $(-1,0)$ and $(0,1)$

② $\lim_{x \to +\infty} x^3 - x^2 - x + 1 = +\infty$

$\lim_{x \to -\infty} x^3 - x^2 - x + 1 = -\infty$

③ N/A → none

④ $f'(x) = 3x^2 - 2x - 1$

$f'(x) = 0$ when $0 = 3x^2 - 2x - 1$

$$0 = (3x+1)(x-1)$$
$$3x+1 = 0 \quad , \quad x-1 = 0$$
$$x = -\tfrac{1}{3} \quad\quad x = 1$$

$f(-\frac{1}{3}) = \frac{32}{27}$ or $1\frac{5}{27}$ $\quad f(1) = 0$

$\therefore$ critical points at $(-\frac{1}{3}, \frac{32}{27})$ and $(1,0)$.

⑤ $f'(x) = 3x^2 - 2x - 1$

$f''(x) = 6x - 2$

$f''(x) = 0 \rightarrow 0 = 6x - 2$ $\quad f(\frac{1}{3}) = \frac{16}{27}$

$6x = 2$

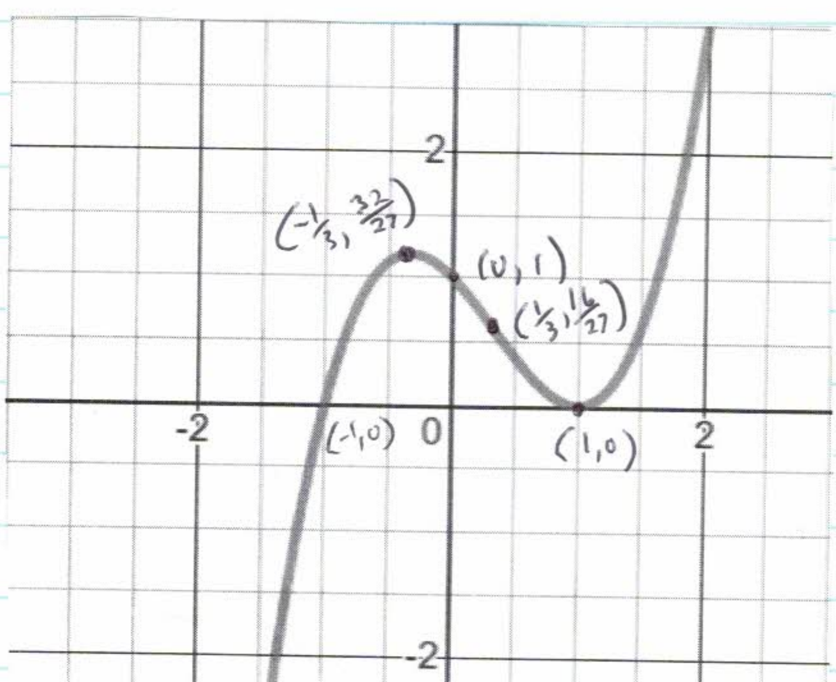
$x = \frac{1}{3}$

⑥ $\therefore$ $(-\frac{1}{3}, \frac{32}{27})$ and $(1,0)$ are <u>both</u> turning points.

(since 2nd derivative only zero at $x = \frac{1}{3}$)

$(\frac{1}{3}, \frac{16}{27})$ is a point of inflection.

⑦



note that

$f''(1) = 6(1) - 2$

$= 4$

> 0

$f''(-\frac{1}{3}) = 6(-\frac{1}{3}) - 2$

$= -4$

< 0

$\boxed{\text{Example 2}}$

$$f(x) = \frac{x}{x^2+1}$$

① intercepts $\quad f(0) = 0 \quad (0,0)$

② $$\lim_{x \to +\infty} \frac{x}{x^2+1} = 0^+ \qquad \frac{1}{x}$$

$$\lim_{x \to -\infty} \frac{x}{x^2+1} = 0^-$$

③ no vertical asymptotes since $x^2+1 \neq 0$

④ $$f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{-x^2+1}{(x^2+1)^2}$$

$$f'(x) = 0 \to -x^2+1 = 0 \qquad f(1) = \tfrac{1}{2}$$

$$x^2 = 1 \qquad f(-1) = -\tfrac{1}{2}$$

$$x = \pm 1$$

critical points $(1, \frac{1}{2})$ and $(-1, -\frac{1}{2})$

(5) $$f''(x) = \frac{-2x(x^2+1)^2 - (-x^2+1)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$f''(x) = \frac{-2x(x^2+1)^2 - 4x(-x^2+1)(x^2+1)}{(x^2+1)^4}$$

$$f''(x) = \frac{-2x(x^2+1)\left((x^2+1) + 2(-x^2+1)\right)}{(x^2+1)^4}$$

$$f''(x) = \frac{-2x(x^2+1)(-x^2+3)}{(x^2+1)^4}$$

$$f''(x) = \frac{-2x(-x^2+3)}{(x^2+1)^3}$$

$f''(x) = 0$ when $x = 0$ and $-x^2+3 = 0$

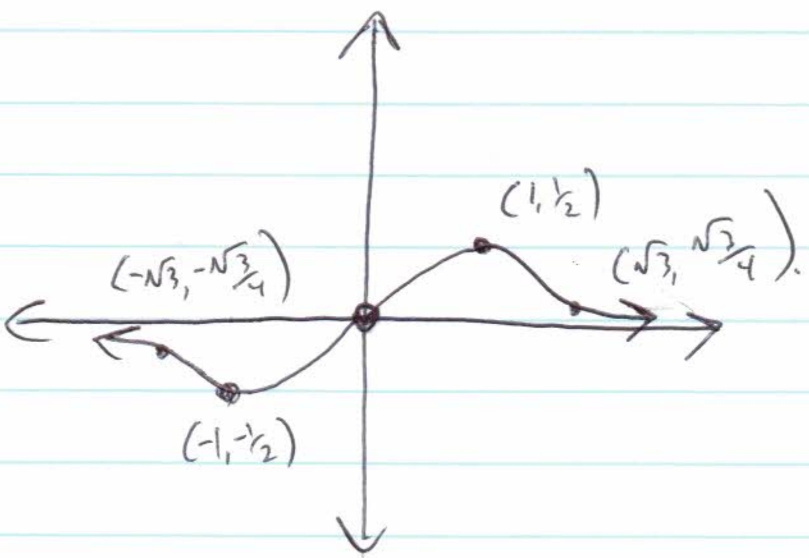
$x^2 = 3$

$x = \pm\sqrt{3}$

$f(0)=0 \qquad f(\sqrt{3})=\dfrac{\sqrt{3}}{4} \qquad f(-\sqrt{3})=\dfrac{-\sqrt{3}}{4}.$

(6) $\therefore$ $(1,\frac{1}{2})$ and $(-1,-\frac{1}{2})$ are turning points and $(0,0)$ $(\sqrt{3},\sqrt{3}/4)$ and $(-\sqrt{3},-\sqrt{3}/4)$ are points of inflection

(7)



Example 3 $\quad f(x) = \dfrac{x^3}{x^3+1}$

① $f(0) = 0$ (only intercept)

② $$\lim_{x \to +\infty} \frac{x^3}{x^3+1} = 1^-$$

$$\lim_{x \to -\infty} \frac{x^3}{x^3+1} = 1^+$$

$\therefore$ $y = 1$ a horizontal asymptote.

③ vertical asymptote at $x = -1$

④ $$f'(x) = \frac{3x^2(x^3+1) - x^3(3x^2)}{(x^3+1)^2}$$

$$f'(x) = \frac{3x^5 + 3x^2 - 3x^5}{(x^3+1)^2}$$

$$f'(x) = \frac{3x^2}{(x^3+1)^2} \qquad f'(x) = 0 \rightarrow x = 0$$

only critical point at $(0, 0)$.

⑤ $$f''(x) = \frac{6x(x^3+1)^2 - 3x^2(2)(x^3+1)(3x^2)}{(x^3+1)^4}$$

$$f''(x) = \frac{6x(x^3+1)^2 - 18x^4(x^3+1)}{(x^3+1)^4}$$

$$f''(x) = \frac{6x(x^3+1)\,(x^3+1-3x^3)}{(x^3+1)^4}$$

$$f''(x) = \frac{6x\,(1-2x^3)}{(x^3+1)^3}$$

$f''(x) = 0$ at $x = 0$ and $(1-2x^3) = 0$

$2x^3 = 1$

$x^3 = 1/2$

$x = \sqrt[3]{1/2}$

$x = \frac{1}{\sqrt[3]{2}}$

$f(0) = 0$

$$f\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{1/2}{\frac{1}{2}+1}$$

$$= \frac{1/2}{3/2} = \frac{1}{2} * \frac{2}{3}$$

$$= 1/3$$

⑥ $\therefore \left(\frac{1}{\sqrt[3]{2}}, 1/3\right)$ is a point of inflection.

but $f'(0) = f''(0) = 0$ so 2nd derivative test fails!

Need to figure out what is going on at $(0,0)$.