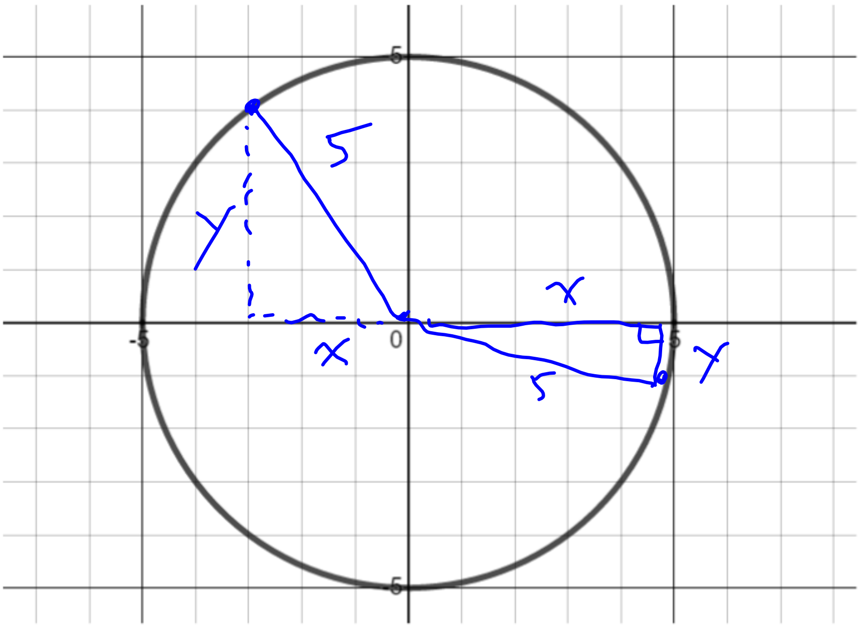
Implicit Differentiation

Consider the example of finding the derivative of a circle.



$x^2 + y^2 = 5^2$

$x^2 + y^2 = 25$

$x^2 + y^2 = r^2$

How do we take the derivative of an equation defined *implicitly?*

Equations Defined Explicitly: (Think "clearly defined")

Equations are defined explicitly are written in the form y = f(x) and are the only type we have dealt with so far in this course.

Examples: y = x^2, $y = \sqrt{x^2} - 6$ y = 3x + 1 etc.

$f(x) = \sqrt[3]{x^2 - 1}$

Equations Defined Implicitly:

Equations can be defined implicitly (y is not given explicitly in terms of x).

Examples: $x^2 + y^2 = 25,\ y^5 + x^2y - 2x^2 = -1,$ etc..

Equations like the ones above may or may not be rewritten explicitly. Often they are not functions.

$y^2 = 25 - x^2 \longrightarrow y = \pm\sqrt{25 - x^2}$

Implicit equations have instantaneous rate of change (derivatives) and tangent lines, so we want to be able to differentiate them as well. This process is called implicit differentiation.

Examples

Find $\frac{dy}{dx}$ for the circle defined by the equation above. Find the equation of the tangent to the circle at the point (3,4).

$$x^2 + y^2 = 25$$

$$2x + \boxed{2y \cdot \frac{dy}{dx}} = 0$$

$$2y \cdot \frac{dy}{dx} = -2x$$

$$y \cdot \frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

at (3,4)

$$\frac{dy}{dx} = -\frac{3}{4}$$

$$-\frac{3}{4} = \frac{y-4}{x+3}$$

$$-3x-9 = 4y-16$$

$$\boxed{3x+4y-7=0}$$

$$x \to u \to y \qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$x^2 + y^2 = 25$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$x \to y \to y^2$$

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \cdot \frac{dy}{dx}$$

Find $\frac{dy}{dx}$ for the curved defined by $\frac{x^2}{4} + y^2 = 1$

$$x^2 + y^2 = 1$$

$$\left(\frac{x}{2}\right)^2 + y^2 = 1$$

$$\frac{2x}{4} + 2y \cdot \frac{dy}{dx} = 0$$

$$2y \cdot \frac{dy}{dx} = -\frac{1}{2}x$$

$$\frac{dy}{dx} = \frac{-x}{4y}$$

Sometimes implicit differentiation is easier even if the function is defined explicitly.

Example: Find the slope of the tangent to the curve $y = \sqrt[3]{x^2+4}$ at the point where x = 2.

$$y^3 = x^2 + 4$$

$$3y^2 \cdot \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2}$$

at $x = 2$

$$y = \sqrt[3]{2^2+4}$$

$$y = 2$$

$$\frac{dy}{dx} = \frac{2(2)}{3(2)^2}$$

$$= \frac{4}{12} = \frac{1}{3}$$

Implicit differentiation can make finding a derivative easier. However, for equation such as $y^5 + x^2y - 2x^2 = -1$, it is the only way to find a derivative (this equation cannot be written explicitly).

$$y^5 + x^2y - 2x^2 = -1$$

$y = \_$

$$5y^4 \cdot \frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} - 4x = 0$$

$$5y^4\frac{dy}{dx} + x^2\frac{dy}{dx} = 4x - 2xy$$

$$\frac{dy}{dx}(5y^4 + x^2) = 4x - 2xy$$

$$\frac{dy}{dx} = \frac{4x - 2xy}{5y^4 + x^2}$$

$$(x^2+y^2)^2 = 5x - x^3y^2$$

$$2(x^2+y^2)\left(2x+2y\cdot\frac{dy}{dx}\right)$$

$$= 5 - \left(3x^2y^2 + x^3\,2y\cdot\frac{dy}{dx}\right)$$

$$(2x^2+2y^2)\left(2x+2y\cdot\frac{dy}{dx}\right)$$

$$= 5 - 3x^2y^2 + x^3\,2y\cdot\frac{dy}{dx}$$

$$4x^3 + 4x^2y\frac{dy}{dx} + 4xy^2 + 4y^3\frac{dy}{dx}$$

$$= 5 - 3x^2y^2 + x^3\,2y\cdot\frac{dy}{dx}$$

$$4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} - x^3\,2y\frac{dy}{dx}$$

$$= 5 - 3x^2y^2 - 4x^3 - 4xy^2$$

$$\frac{dy}{dx} = \frac{5-3x^2y^2-4x^3-4xy^2}{4x^2y+4y^3-2x^3y}$$