$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Chain Rule

$y = u^4$

Consider taking the derivative of the function:

$y = (x^2 - 2x)^4$

$\frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{dx} = 4(x^2-2x)^3 \; (2x-2)$

There really is no such rule as the "power of a function" rule. It is actually the power rule used with what is referred to as the ***chain rule.***

$x \to u \to y$

The chain rule is as follows:

$y \leftarrow u$

If *y* is a function of *u* and *u* is a function of *x* then:

$u \leftarrow x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

We can rewrite the example above and show how the chain rule (with the power rule) gives us the derivative:

$y = (x^2-2x)^4$ let $u = x^2-2x$

$y = u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 4u^3 \; (2x-2)$$

$$= 4(x^2-2x)(2x-2$$

In fact, we can combine the power rule with the chain rule and write:

$y = u^n$

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$$

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We can also use the chain rule multiple times, with any variable names we like.

$x \to u \to y \to z$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \qquad \frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{du} \times \frac{du}{dx}$$

$t \to r \to A$

$$\frac{dy}{dx} = \frac{dy}{dr} \times \frac{\cancel{du}}{dx} \quad dr \qquad \frac{ds}{dx} \quad \frac{\cancel{dx}}{ds} = \frac{ds}{dt} \times \frac{dt}{dx}$$

$x \to t \to s$

The chain rule is more difficult to understand in prime notation. But it is defined as:

If $h(x) = f[g(x)]$ then $h'(x) = f'[g(x)]g'(x)$

$(f \circ g)(x)$

The chain rule does not immediately give us any new functions we can take the derivative of. (Since we have already been using the chain rule without knowing it).

$f(x) = \sqrt{3x^2 - x}$

$$f(x) = (3x^2 - x)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(3x^2 - x)^{-1/2}(6x - 1)$$

$$= \frac{6x - 1}{2\sqrt{3x^2 - x}}$$

We could use the chain rule to simplify the following problem:

Find the slope of the tangent to the graph of

$$y = \left[4(3-x^2) + \frac{3}{(3-x^2)}\right]^5 + \left[4(3-x^2) + \frac{3}{(3-x^2)}\right]^3 \text{ at x = 2}$$

Let $v = 3 - x^2$

$u = 4v + 3v^{-1}$

$y = u^5 + u^3$

$x \to v \to u \to y$

$x = 2,\ v = -1$

$u = -7$

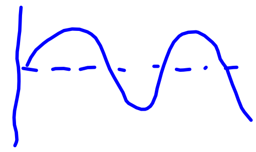
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= (5u^4 + 3u^2)(4 - 3v^{-2})(-2x)$$

$$= (5(-7)^4 + 3(-7)^2)(4 - 3(-1)^{-2})(-2(2))$$

$$= -48\,608$$

Here is a new derivative rule for you: $\frac{d}{dx}(\sin x) = \cos x$



The chain rule will now allow us to find derivatives of all sorts of trigonometric functions:

$y = \sin(3x^2 - 4x)$

$$\frac{dy}{dx} = \cos(3x^2 - 4x)\ (6x-4)$$

$$= (6x-4) \cdot \cos(3x^2 - 4x)$$

$x^{\frac{1}{2}}$

$f(x) = 10 \sin \sqrt{x}$

$$f'(x) = 10 \cos\sqrt{x} \left(\frac{1}{2} x^{-\frac{1}{2}}\right)$$

$$= \frac{5 \cos\sqrt{x}}{\sqrt{x}}$$

Can we prove the chain rule?

Let *y* be a function of *u* and *u* be a function of *x*:

$x \to u \to y$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \frac{\Delta u}{\Delta u}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

as $\Delta x \to 0$, $\Delta u \to 0$

assume $\Delta x \to 0$ $\Delta u \neq 0$

$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$\frac{dy}{dy} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \square$$

$y = \sin x \qquad y' = \cos x$

$$y = \sin^2(3x-1)$$

$$y = [\sin(3x-1)]^2$$

$$\frac{dy}{dx} = 2[\sin(3x-1)] \cos(3x-1)(3)$$

$$= 6 \sin(3x-1) \cos(3x-1)$$