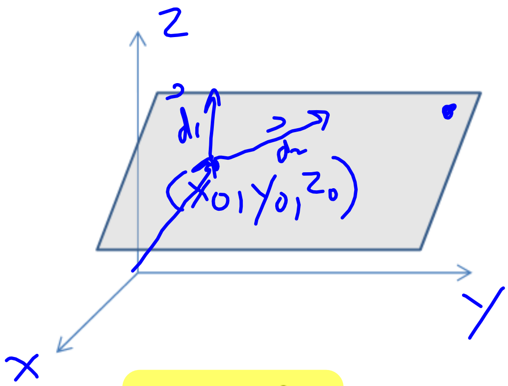
MCV4U

Equations of Planes

Planes are flat 2-D surfaces. The Cartesian (xy-plane) is an example of a plane. We often draw planes as parallelograms on a flat surface, even though the planes extend in all directions.

Using a vector equation of a line, we define a line by establishing position on a line (initial point) then adding a scalar multiple of a direction vector to arrive at new points. We can define a plane using a vector equation similar to a line.



The **vector equation** of a plane is given by:

$(x, y, z) = (x_0, y_0, z_0) + t(a_1, b_1, c_1) + s(a_2, b_2, c_2)$ where $s, t \in R$.

direction vectors

We then can define **parametric equations** of plane of the form:

$x = x_0 + sa_1 + ta_2$

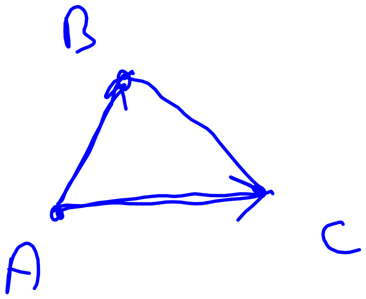
$y = y_0 + sb_1 + tb_2$

$z = z_0 + sc_1 + tc_2$

where $s, t \in R$.

Any three non-collinear points define a plane.

Find the vector and parametric equations of the plane formed by the points A(1, 2, 3), B(4, 5, -3) and C(0,0,6).



$\vec{AB} = (3, 3, -6)$ $\quad \vec{d_1} = (1, 1, -2)$

$\vec{AC} = (-1, -2, 3)$ $\quad \vec{d_2} = (-1, -2, 3)$

$$\vec{r} = (1, 2, 3) + t(-1, -2, 3) + s(1, 1, -2)$$

$s, t \in R$

$$\left.\begin{aligned} x &= 1 - t + s \\ y &= 2 - 2t + s \\ z &= 3 + 3t - 2s \end{aligned}\right\}$$

<u>point</u> $s = 1, \ t = 2$

$x = 1 - 2 + 1 = 0$

$y = 2 - 4 + 1 = -1$

$z = 3 + 6 - 2 = 7$

$(0, -1, 7)$

Does the point (-7, 3, -1) lie in the plane formed above?

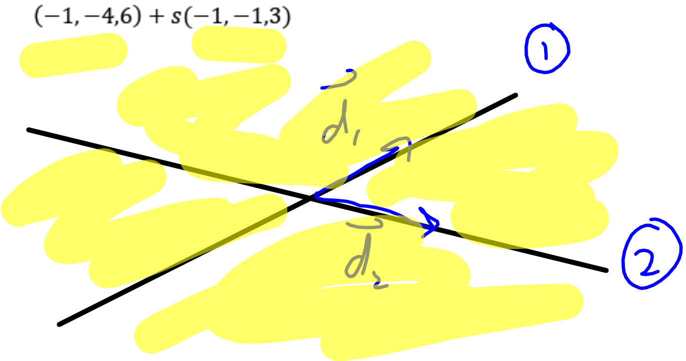
$$\left.\begin{aligned} x &= 1 - t + s \\ y &= 2 - 2t + s \\ z &= 3 + 3t - 2s \end{aligned}\right\}$$

3 eqns
2 unknowns

$$-7 = 1 - t + s$$
$$3 = 2 - 2t + s$$
$$-1 = 3 + 3t - 2s$$

no solution

Find the equation of a plane that containing two intersecting lines: $\vec{r}_1 = (4,7,3) + t(1,4,3)$ and $\vec{r}_2 = (-1,-4,6) + s(-1,-1,3)$



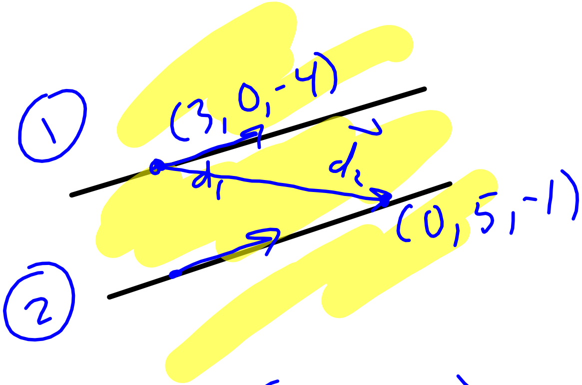
$$(x,y,z) = (4,7,3) + m(1,4,3) + n(-1,-1,3)$$

$$m, n \in R$$

Find the equation of the plane containing the lines $\frac{x-3}{2} = \frac{y}{-3} = \frac{z+4}{3}$ and $x = 2t, y = -3t + 5, \quad z = 3t - 1$.

$\vec{d}_1 = (2,-3,3)$ $\cancel{\vec{d}_2} = (2,-3,3)$

$\vec{d}_2 = (-3, 5, 3)$



$$(x,y,z) = (3,0,-4) + s(2,-3,3) + r(-3,5,3)$$

$s, r \in R$

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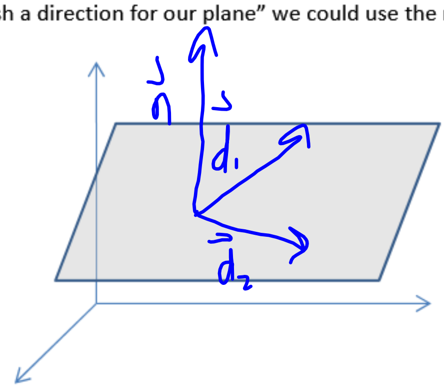
Instead of using 2 direction vectors to "establish a direction for our plane" we could use the **normal to the plane.**



Example Find a normal to the plane $\vec{r} = (1,2,9) + t(-1,0,1) + s(1,-3,4)$.

$$\begin{array}{cccccc} -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & -3 & 4 & 1 & -3 & 4 \end{array}$$

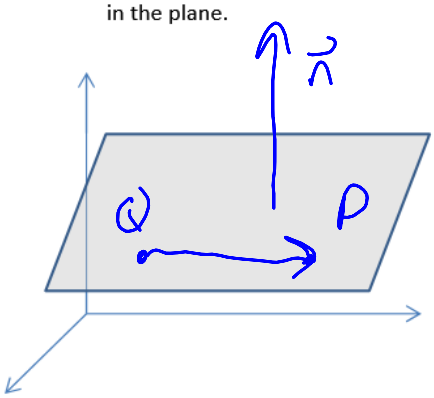
$$(-1,0,1) \times (1,-3,4)$$

$$= (3,5,3) \qquad \vec{n} = (3,5,3)$$

Is the normal unique?

The normal to a plane will be perpendicular to any vector in the plane. We can use this fact to derive the scalar equation to a plane.

Let P(x,y,z) be any point in the plane with normal (A,B,C). Let Q (x_0,y_0,z_0) be a particular point (given point) in the plane.



$$\vec{QP} \cdot \vec{n} = 0$$

$$(x-x_0, y-y_0, z-z_0) \cdot (A,B,C) = 0$$

$$Ax - Ax_0 + By - By_0 + Cz - Cz_0 = 0$$

$$Ax + By + Cz \underbrace{- Ax_0 - By_0 - Cz_0}_{D} = 0$$

$$Ax + By + Cz + D = 0$$

What is the scalar equation for the plane from the example on the previous page?

$\vec{r} = (1,2,9) + t(-1,0,1) + s(1,-3,4).$

$\vec{n} = (3,5,3)$

$3x + 5y + 3z + D = 0$

sub. in $(1,2,9)$

$3(1) + 5(2) + 3(9) + D = 0$

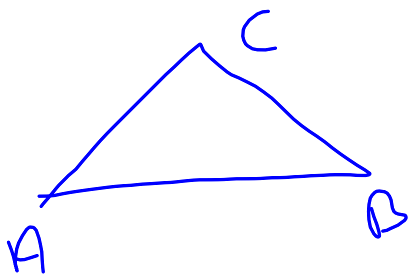
$40 + D = 0$

$D = -40$

$\boxed{3x+5y+3z-40=0}$

The scalar equation is unique.

Example: Find the scalar equation of the plane containing the 3 points A(1,1,1), B(2, 0, -1) and C(0, -1, 4).



$\vec{AB} = (1,-1,-2)$

$\vec{AC} = (-1,-2,3)$

$\vec{AB} \times \vec{AC} = (-7,-1,-3)$

let $\vec{n} = (7,1,3)$

$7x + y + 3z + D = 0$

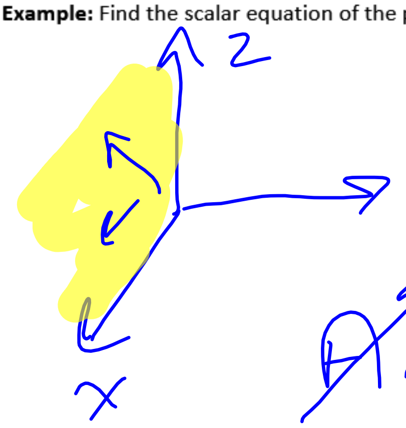
sub in $(1,1,1)$

$7 + 1 + 3 + D = 0$

$D = -11$

$7x + y + 3z - 11 = 0$

Example: Find the scalar equation of the plane containing the point (4, 3, -2) and is parallel to the xz-plane.



$\vec{n} = (0, 1, 0)$

$\cancel{A}^{\,0}x + By + \cancel{C}^{\,0}z + D = 0$

$y + D = 0$

sub $(4, 3, -2)$

$3 + D = 0$

$D = -3$

$\boxed{y - 3 = 0}$ $\quad y = 3$

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