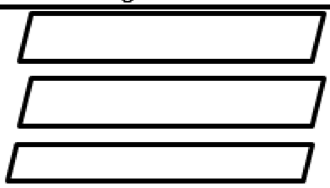
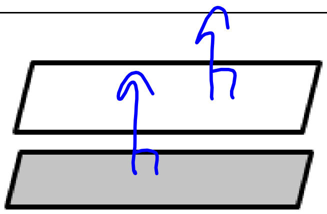
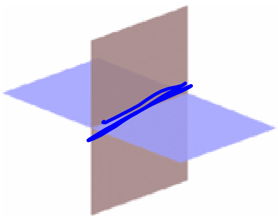
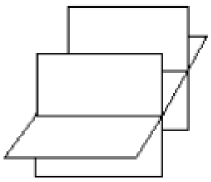
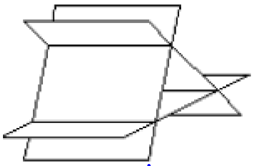
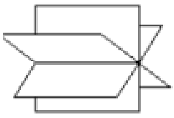
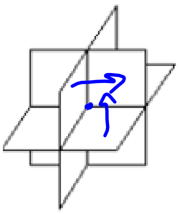
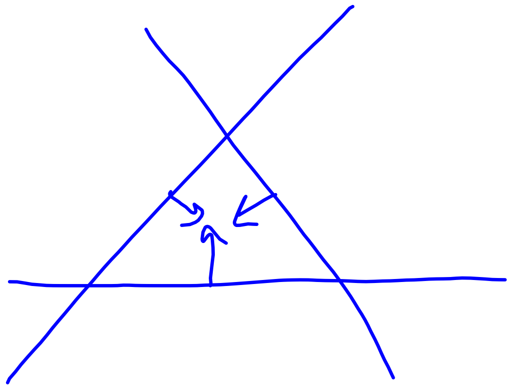
$$\begin{aligned} &(1)\quad 2x + y - z = 5 \\ &(2)\quad x + y + 2z = 7 \\ &(3)\quad 4x - y - z = 0 \end{aligned}$$

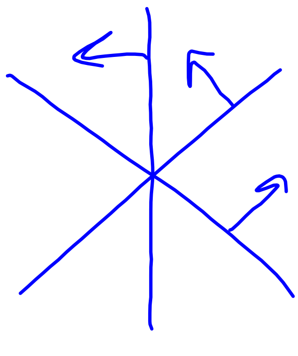
Today we will look at all of the geometric possibilities for the intersection of three planes in space.
We will summarize these possibilities in the table below. Let the normal for the planes be $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$.

| Diagram | Intersection | Geometric Properties |
|---|---|---|
|  | no | planes distinct |
|  | no | $\vec{n}_1, \vec{n}_2, \vec{n}_3$ are collinear<br>2 planes coincident |
|  | plane | all 3 planes coincident |

| | | |
|---|---|---|
|  | line | 2 coincident planes<br>2 normals collinear |
|  | no | 2 normals collinear |

| | | |
|---|---|---|
|  triangular prism | no | no collinear normals<br>normals coplanar<br>$\boxed{(\vec{n}_1 \times \vec{n}_2)\cdot \vec{n}_3 = 0}$ |
|  | line | normals coplanar |
|  | point | normals are non coplanar<br>$(\vec{n}_1 \times \vec{n}_2)\cdot \vec{n}_3 \neq 0$ |





Can you determine the intersection of the following planes without solving?

$$2x + y + z + 4 = 0$$
$$3y - 2z - 2 = 0$$
$$3x + y + 2z + 7 = 0$$

coplanar?

$\vec{n}_1 = (2, 1, 1)$

$\vec{n}_2 = (0, 3, -2)$

$\vec{n}_3 = (3, 1, 2)$

$(2, 1, 1) \times (0, -3, -2) \cdot (3, 1, 2)$

$= (-5, 4, 6) \cdot (3, 1, 2)$

$= -15 + 4 + 12$

$= 1 \neq 0$

intersection is a point.

$x - 2y + 3z = 9$

$x + y - z = 4$

$2x - 4y + 6z = 5$

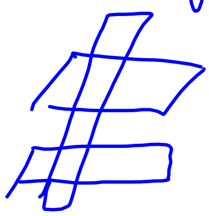
$\vec{n}_1 = (1, -2, 3)$

$\vec{n}_2 = (1, 1, -1)$

$\vec{n}_3 = (2, -4, 6)$

$\vec{n}_3 = 2\vec{n}_1$

no intersection



$x - y + 4z = 5$
$3x + y + z = -2$
$5x - y + 9z = 1$

coplanar?

$(1, -1, 4) \times (3, 1, 1) \cdot (5, -1, 9)$
$= 0$

~~triangular prism~~ or line
(no intersection)

need to solve!

It is a good time to take a break and try to solve question #1 from the assigned problems.

Now lets look at the **Algebra.**

How do we solve for the intersection in the first example above?

point

**2x + y + z + 4 = 0**
**3y – 2z – 2 = 0**
**3x + y + 2z + 7 = 0**

$$\begin{aligned} 2x + y + z &= -4 \\ 3y - 2z &= 2 \\ 3x + y + 2z &= -7 \end{aligned}$$

Solving a linear system with 3 equations and 3 unknowns is a difficult process. We usually take advantage of what is referred to as a **matrix.** A matrix is just a rectangular array of numbers.

( ) | |

**Matrix Examples**

$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & 4 \\ 1 & 3 & 0 \end{bmatrix}$ 3×3 $\quad \begin{bmatrix} 2 & 0 & 3 \\ 4 & -1 & 1 \end{bmatrix}$ 2×3 $\quad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ 1×3 $\quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 3×1 $\quad \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}$ 2×2

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3×3 $\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ 2×3

A system of equations can be represented by an **augmented matrix.** We begin by writing the augmented matrix for the system above. It has dimensions **3 x 4**.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & -4 \\ 0 & 3 & -2 & 2 \\ 3 & 1 & 2 & -7 \end{array}\right]$$

The augmented matrix then represents a linear system of equations and so the following operations can be performed on the rows:

<u>**Elementary Row Operations**</u>

1. Interchange 2 rows.
2. Multiply (or divide) a row by a non-zero constant.
3. Replace a row with the sum (or difference) of itself and a multiple of another row.

These operations are similar to those used in the **elimination method,** with the matrix be used to organize our work. Similar to the elimination method we wish to eliminate certain entries – by making them equal to zero.

We can use row operations to solve the first linear system/intersection of 3 planes example.

**2x + y + z = -4**

**3y – 2z = 2**

**3x + y + 2z = -7**

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & -4 \\ 0 & 3 & -2 & 2 \\ 3 & 1 & 2 & -7 \end{array}\right] \xrightarrow{3r1-2r3} \left[\begin{array}{ccc|c} 2 & 1 & 1 & -4 \\ 0 & 3 & -2 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{r2-3r3} \left[\begin{array}{ccc|c} 2 & 1 & 1 & -4 \\ 0 & 3 & -2 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$\boxed{z = -4}$$

The process we just did is called ***Gaussian elimination.*** We ended up with a matrix in row-echelon form. We can then **back substitute** and solve for *x, y* and *z.*

$$3y - 2z = 2$$
$$3y - 2(-4) = 2$$
$$3y + 8 = 2$$
$$3y = -6$$
$$\boxed{y = -2}$$

$$2x + y + z = -4$$
$$2x + (-2) + (-4) = -4$$
$$2x - 6 = -4$$
$$\boxed{x = 1}$$

$$(1, -2, -4)$$

We could have kept going with our matrix operations. ***Guass-Jordan Elimination*** is the process of putting a matrix in **reduced row-echelon** form.

So, instead of **back-substitution** we could have put our matrix in **reduced row-echelon form** by making everything but the diagonals entries in the matrix equal to zero.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & -4 \\ 0 & 3 & -2 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right] \xrightarrow{r2-3r1} \left[\begin{array}{ccc|c} -6 & 0 & -5 & 14 \\ 0 & 3 & -2 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$\xrightarrow{5r3+r1} \left[\begin{array}{ccc|c} -6 & 0 & 0 & 6 \\ 0 & 3 & -2 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right] \xrightarrow{2r3+r2} \left[\begin{array}{ccc|c} -6 & 0 & 0 & 6 \\ 0 & 3 & 0 & -6 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$\xrightarrow[\frac{1}{3}r2]{-\frac{1}{6}r1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right] \qquad \begin{array}{l} x = 1 \\ y = -2 \\ z = -4 \end{array}$$

Strategy for reducing rows:

For a system of 3 equations and 3 unknowns, carry out operations that will turn each number into zero in the ordered shown below. (note this is only a ***coefficient matrix***)

$$\begin{bmatrix} * & \#4 & \#5 \\ \#1 & * & \#6 \\ \#2 & \#3 & * \end{bmatrix}$$

1 – 3 ← use back substitution

Try a similar example. Find the solution to the following system. Use either Gaussian or Gauss-Jordan elimination, your choice!

$$x - 3y - 2z = -9$$
$$2x - 5y + z = 3$$
$$-3x + 6y + 2z = 8$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -9 \\ 2 & -5 & 1 & 3 \\ -3 & 6 & 2 & 8 \end{array}\right] \xrightarrow{2r1-r2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & -9 \\ 0 & -1 & -5 & -21 \\ -3 & 6 & 2 & 8 \end{array}\right]$$

$$\xrightarrow{3r1+r3} \left[\begin{array}{ccc|c} 1 & -3 & -2 & -9 \\ 0 & -1 & -5 & -21 \\ 0 & -3 & -4 & -19 \end{array}\right] \xrightarrow{3r2-r3} \left[\begin{array}{ccc|c} 1 & -3 & -2 & -9 \\ 0 & -1 & -5 & -21 \\ 0 & 0 & -11 & -44 \end{array}\right]$$

$$\xrightarrow{3r2-r1} \left[\begin{array}{ccc|c} -1 & 0 & -13 & -54 \\ 0 & -1 & -5 & -21 \\ 0 & 0 & -11 & -44 \end{array}\right] \xrightarrow[-r3]{-r1,\ -r2} \left[\begin{array}{ccc|c} 1 & 0 & 13 & 54 \\ 0 & 1 & 5 & 21 \\ 0 & 0 & 11 & 44 \end{array}\right]$$

$$\xrightarrow{13r3-11r1} \left[\begin{array}{ccc|c} -11 & 0 & 0 & -22 \\ 0 & 1 & 5 & 21 \\ 0 & 0 & 11 & 44 \end{array}\right] \xrightarrow[5r3-11r2]{-\frac{1}{11}r1,\ \frac{1}{11}r3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -11 & 0 & -11 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{11}r2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

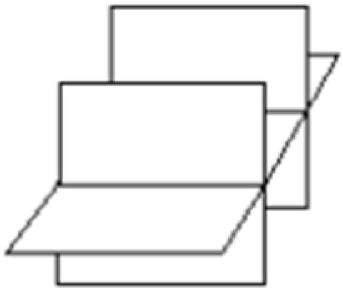
$x = 2$

$y = 1$

$z = 4$

We already determined that there will be no intersection for the planes (or solutions to the linear system) below. We will use a matrix to see what happens.

$x - 2y + 3z = 9$
$x + y - z = 4$
$2x - 4y + 6z = 5$



$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 1 & 1 & -1 & 4 \\ 2 & -4 & 6 & 5 \end{array}\right] \xrightarrow{r1-r2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & -3 & 4 & 5 \\ 2 & -4 & 6 & 5 \end{array}\right]$$

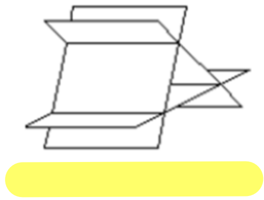
$$\xrightarrow{2r1-r3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & -3 & 4 & 5 \\ 0 & 0 & 0 & 13 \end{array}\right] \qquad 0 \neq 13$$

no solution

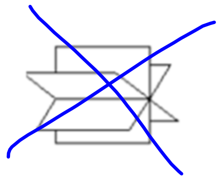
In the third example we could not tell what type of intersection exists.

x – y + 4z = 5
3x + y + z = -2
5x – y + 9z = 1

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 5 \\ 3 & 1 & 1 & -2 \\ 5 & -1 & 9 & 1 \end{array}\right] \xrightarrow{3r1-r2} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 5 \\ 0 & -4 & 11 & 17 \\ 5 & -1 & 9 & 1 \end{array}\right]$$



$$\xrightarrow{5r1-r3} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 5 \\ 0 & -4 & 11 & 17 \\ 0 & -4 & 11 & 24 \end{array}\right]$$



$$\xrightarrow{r2-r3} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 5 \\ 0 & -4 & 11 & 17 \\ 0 & 0 & 0 & -7 \end{array}\right]$$

$0 \neq -7$

no solution

Solve the following.

$$x + y + 2z = -2$$
$$3x - y + 14z = 6$$
$$x + 2y = -5$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 1 & 2 & 0 & -5 \end{array}\right] \xrightarrow{3r1-r2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 4 & -8 & -12 \\ 1 & 2 & 0 & -5 \end{array}\right]$$

$$\xrightarrow[\frac{1}{4}r2]{r1-r3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & 2 & 3 \end{array}\right] \xrightarrow{r2+r3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{r2-r1} \left[\begin{array}{ccc|c} -1 & 0 & -4 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$0z = 0$

let $z = t$

$y - 2z = -3$

$y - 2t = -3$

$y = -3 + 2t$

$-x - 4z = -1$

$-x - 4t = -1$

$-x = -1 + 4t$

$x = 1 - 4t$

line

$$\boxed{\begin{array}{l} x = 1 - 4t \\ y = -3 + 2t \\ z = t \qquad t \in R \end{array}}$$