Intersection of 3 Planes – Solutions

1\. a) $\vec{n}_1 = (1, 2, -1)$

$\vec{n}_2 = (1, -5, 4)$

$\vec{n}_3 = (3, -1, 2)$

Clearly normals are not collinear.

Coplanar? $(1,2,-1) \times (1,-5,4) \cdot (3,-1,2)$

$= (3, -5, -7) \cdot (3, -1, 2)$

$= 9 + 5 - 14$

$= 0$ yes, they are coplanar.

$\begin{array}{cccccc} 1 & 2 & -1 & 1 & 2 & -1 \\ 1 & -5 & 4 & 1 & -5 & 4 \end{array}$

$\therefore$ intersection is a line

or there is no intersection

(not possible to tell).

b) $\vec{n}_1 = (1, 2, 1)$

$\vec{n}_2 = (2, -1, 1)$

$\vec{n}_3 = (3, 1, -2)$

normals are not collinear

coplanar? $(1, 2, 1) \times (2, -1, 1) \cdot (3, 1, -2)$

$= (3, 1, -5) \cdot (3, 1, -2)$

$\begin{matrix} 1 & 2 & 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 & -1 & 2 \end{matrix}$ $= 9 + 1 + 10$

$= 20$. not coplanar.

$\therefore$ intersection is a point.

c) $\vec{n}_1 = (1, -3, 0)$

$\vec{n}_2 = (2, 1, -1)$

$\vec{n}_3 = (4, 2, -2)$

$\vec{n}_2$ and $\vec{n}_3$ are collinear.

But the planes are all distinct.

$\therefore$ no intersection.

d) $\vec{n}_1 = (-2, 4, 6)$

$\vec{n}_2 = (4, -8, -12)$

$\vec{n}_3 = (1, -2, -3)$

All 3 normals are collinear.

Looking at equations all can be reduced to:

$x - 2y - 3z = 1$ All 3 planes are the same!

$\therefore$ intersection is the plane.

2.

a) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 2 & -1 & 1 & 5 \\ 3 & 1 & -2 & 1 \end{array}\right] \xrightarrow{2r1-r2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 0 & 5 & 1 & 19 \\ 3 & 1 & -2 & 1 \end{array}\right]$

$\xrightarrow{3r1-r3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 0 & 5 & 1 & 19 \\ 0 & 5 & 5 & 35 \end{array}\right] \xrightarrow{r2-r3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 0 & 5 & 1 & 19 \\ 0 & 0 & -4 & -16 \end{array}\right]$

$-4z = -16 \rightarrow z = 4$

$5y + z = 19$

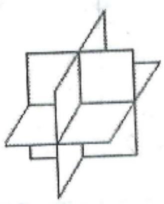
$5y + 4 = 19$

$y = 3$

$x + 2(3) + 4 = 12$

$x = 2.$

$\therefore$ Point of intersection at $(2, 3, 4)$.

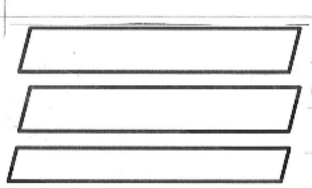


b) $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & -2 & 4 & 7 \\ 3 & -3 & 6 & 11 \end{array}\right]$ note $\vec{n}_1 = \frac{1}{2}\vec{n}_2$

$\vec{n}_1 = \frac{1}{3}\vec{n}_2$

But planes are distinct. No solution.

$\xrightarrow{2r1-r2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & -3 \\ 3 & -3 & 6 & 11 \end{array}\right]$ $0z \neq -3$

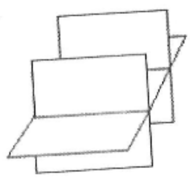


c) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 2 & 2 & -4 & 6 \\ 1 & 1 & -2 & 3 \end{array}\right]$ $\vec{n}_2 = 2\vec{n}_3$ and planes are not coincident.

No intersection.

$\xrightarrow{2r1-r2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 0 & 2 & 4 \\ 1 & 1 & -2 & 7 \end{array}\right]$

$\xrightarrow{r1-r3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right]$



$\xrightarrow{r2-r3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{array}\right]$ $0z \neq 2$

d) $\left[\begin{array}{ccc|c} -2 & 4 & 6 & -2 \\ 4 & -8 & -12 & 4 \\ 1 & -2 & -3 & 1 \end{array}\right]$

$\vec{n}_1 = -\frac{1}{2}\vec{n}_2$ and the planes are coincident.

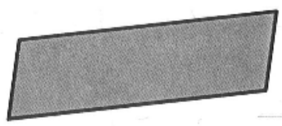
$\vec{n}_1 = -2\vec{n}_1$ and planes are coincident.

Intersection is a plane.

$$\xrightarrow{r1+2r3} \left[\begin{array}{ccc|c} -2 & 4 & 6 & -2 \\ 4 & -8 & -12 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{2r1+r2} \left[\begin{array}{ccc|c} -2 & 4 & 6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}r1} \left[\begin{array}{ccc|c} 1 & -2 & -3 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Intersection is $x - 2y - 3z = 1$.

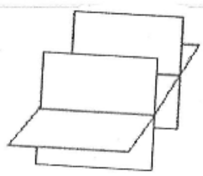


e) $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & -1 & -2 & 5 \\ 3 & 3 & 6 & 5 \end{array}\right]$

$\vec{n}_1 = \frac{1}{3}\vec{n}_2$, planes are //, not coincident.

no intersection.

(similar to part c).



f) $\left[\begin{array}{ccc|c} 1 & 3 & 5 & 10 \\ 2 & 6 & 10 & 18 \\ 1 & 3 & 5 & 9 \end{array}\right]$

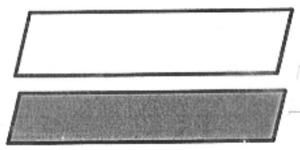
Planes 1 and 2 are // and distinct.

Plane 2 and 3 are coincident

No intersection.

$\xrightarrow{2r1-r2} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 10 \\ 0 & 0 & 0 & 2 \\ 1 & 3 & 5 & 9 \end{array}\right]$

$0z \neq 2$



g) $\left[\begin{array}{ccc|c} 1 & -3 & -2 & 9 \\ 1 & 11 & 5 & -5 \\ 2 & 8 & 3 & 14 \end{array}\right]$

Normals not parallel

$\xrightarrow{r1-r2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 9 \\ 0 & -14 & -7 & 14 \\ 2 & 8 & 3 & 14 \end{array}\right]$

$(1,-3,-2)\times(1,11,5)\cdot(2,8,3)$

$= (7,-7,14)\cdot(2,8,3)$

$= 14-56+42$

$= 0$ Normals are coplanar.

$\xrightarrow{2r1-r3} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 9 \\ 0 & -14 & -7 & 14 \\ 0 & -14 & -7 & 14 \end{array}\right]$

Intersection is either a line, or none (triangular prism). Must solve to find out.

$\xrightarrow{r2-r3} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 9 \\ 0 & -14 & -7 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{-\frac{1}{7}r2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 9 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{3r2+2r1} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 12 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

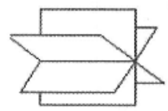
$2x - z = 12 \rightarrow x = \frac{z}{2} + 6$

$2y + z = -2 \rightarrow y = -\frac{z}{2} - 1$

let $z = 2t$

$y = -t - 1$ Intersection is a line.

$x = t + 6$



h) $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 1 & -1 & -4 & -2 \\ 3 & 5 & 12 & 27 \end{array}\right]$ Normals not parallel.

$(1,1,2) \times (1,-1,-4) \cdot (3,5,12)$

$= (-2,6,-2) \cdot (3,5,12)$

$= -6 + 30 - 24$

$= 0$ normals coplanar

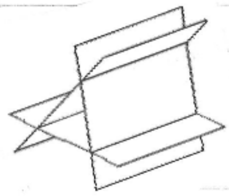
Intersection is a line, or none (triangular prism)

$\xrightarrow{r1-r2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 6 & 8 \\ 3 & 5 & 12 & 27 \end{array}\right]$

$\xrightarrow{3r1-r3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 6 & 8 \\ 0 & -2 & -6 & -9 \end{array}\right]$

$\xrightarrow{r2+r3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 6 & 8 \\ 0 & 0 & 0 & -1 \end{array}\right]$

$0z \neq -1$ no intersection.

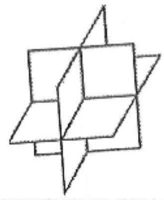


i) $2x+y+z=0$
$x-2y-3z=0$
$3x+2y+4z=0$

normals <u>not</u> parallel

From inspection each plane goes through origin. $(D=0)$

$\therefore$ Intersection is the origin $(0,0,0)$.



3.

$x-2y-z=0$
$x+9y-5z=0$
$kx-y+z=0$

Not possible for these planes to be parallel.

All contain origin, $\therefore$ they do intersect,

$\therefore$ Can not form triangular prism.

If they intersect in a line, normals are coplanar. (otherwise intersect in a point)

$\therefore (1,-2,-1)\times(1,9,-5)\cdot(k,-1,1)=0.$

$=(19,4,11)\cdot(k,-1,1)=0$

$19k - 4 + 11 = 0$

$19k + 7 = 0$

$k = \frac{-7}{19}.$