Euler's Number and Natural Logarithms

In this unit we will be concerned with analyzing exponential functions. Suppose we want to find the derivative of $f(x) = a^x$.

$y = 2^x$

By considering the graph of $f(x) = a^x$, what do we know must be true about the derivative function? (Assume in this case a>1)

always be positive
always be increasing

Then back to first principles…

$f(x) = a^x \qquad f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$

$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$ ← depends on a.

GOAL – Find a base that will create a "convenient" result. Mathematicians sought the base that could make the limit equal to 1. This base was given the symbol e. Often referred to as Euler's Number.

let $e = a$ $\therefore \lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$\therefore \frac{e^h - 1}{h} \to 1$

$\therefore e^h - 1 \to h$

$e^h \to h + 1$

$e \to \sqrt[h]{1+h}$

$$\boxed{\therefore \lim_{h \to 0} (1+h)^{\frac{1}{h}} = e}$$

Euler's Number

$e = \lim_{n \to 0} (1+n)^{\frac{1}{n}}$

OR

$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

$e \doteq (1+0.001)^{\frac{1}{0.001}}$

$\doteq 1.001^{1000} \doteq 2.7169...$

$e \doteq 2.718281828...$

Find the derivative of $f(x) = e^x$

$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$

$\boxed{\frac{d(e^x)}{dx} = e^x} = e^x \lim_{h \to 0} \cancel{\frac{e^h - 1}{h}}^{1} = e^x$

For the graph of $f(x) = e^x$, the slope of the tangent anywhere on the graph is equal to:

the y-value at that point.

Examples: Find the Derivative of the following:

a) $y = e^{-3x}$

$\frac{dy}{dx} = e^{-3x}(-3)$

$\frac{dy}{dx} = -3e^{-3x}$

b) $y = 2 - e^{-x}$

$\frac{dy}{dx} = -e^{-x}(-1)$

$= e^{-x}$

$\frac{dy}{dx} = \frac{1}{e^x}$

$y = e^x$

$\frac{dy}{dx} = e^x$

$y = e^u$

$\frac{dy}{dx} = e^u \cdot \frac{du}{dx}$

Find the critical points of the graph of $y = 2x(e^x) - e^x$

$$\frac{dy}{dx} = 2e^x + 2xe^x - e^x$$

$$\frac{dy}{dx} = 0 = e^x(2 + 2x - 1)$$

$$0 = e^x(2x+1)$$

$$e^x \neq 0 \quad or \quad 2x+1=0$$

$$x = -\tfrac{1}{2}$$

$$x = -\tfrac{1}{2}$$

$$y = 2(-\tfrac{1}{2})e^{-\frac{1}{2}} - e^{-\frac{1}{2}}$$

$$y = -e^{-\frac{1}{2}} - e^{-\frac{1}{2}}$$

$$y = -2e^{-\frac{1}{2}}$$

$$y = \frac{-2}{\sqrt{e}}$$

$$\left(-\tfrac{1}{2}, \frac{-2}{\sqrt{e}}\right)$$

The Natural Logarithm

Recall the meaning of a logarithmic function

$$y = a^x \longleftrightarrow \log_a y = x$$

$$\boxed{\begin{array}{c} 2^5 = 32 \\ \log_2 32 = 5 \end{array}}$$

The function $y = e^x$ is an important function. Its inverse $y = \log_e x$ is so commonly used it has its own notation: $y = \ln x$.

$y = \ln x$ is called the "natural logarithm".

The ln button on your calculator enables you to evaluate the natural logarithm of any number. In fact it can be used instead of the log button in many cases.

$$2^x = 50$$

$$\log 2^x = \log 50$$

$$x \log 2 = \log 50$$

$$x = \frac{\log 50}{\log 2}$$

Example

Evaluate:

ln 3 $\doteq 1.09$ since $e^{1.09} \doteq 3$

ln e = 1 since $e^1 = e$

ln e^2 = 2 since $e^2 = e^2$

$ln\frac{1}{e} = -1$ since $e^? = \frac{1}{e}$

$y = 2^x$

The functions $y = e^x$ and $y = \ln x$ can be used as the foundation for finding the derivatives of all other exponential and logarithmic functions.

The Derivative of $y = \ln x$

If $y = \ln x$ then:

$e^y = x$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y}$

$$\boxed{\frac{dy}{dx} = \frac{1}{x}}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

We often use the chain rule when differentiating with ln x.

Examples Find the derivative of each of the following:

a) $y = \ln 3x^2$

$\frac{dy}{dx} = \frac{1}{3x^2}(6x)$

$= \frac{2}{x}$

b) $y = \ln e^{3x}$

$\frac{dy}{dx} = \frac{1}{\cancel{e^{3x}}} \cancel{e^{3x}}(3)$

$\frac{dy}{dx} = 3$

$y = 3x \ln e$

$y = 3x$

c) $y = x^3 \ln x^2$

$\frac{dy}{dx} = 3x^2 \ln x^2 + x^3 \frac{1}{x^2} \cdot 2x$

$\frac{dy}{dx} = 3x^2 \ln x^2 + 2x^2$

$\frac{dy}{dx} = 6x^2 \ln x + 2x^2$

Working with $y = \ln x$

Can be used to solve for exponential equation with base *e*.

$e^{3x} = 15$

$\ln e^{3x} = \ln 15$

$3x \ln e = \ln 15$

$3x = \frac{\ln 15}{\ln e}$

$x = \frac{\ln 15}{3 \ln e}$ $x = \frac{\ln 15}{3}$

$8e^{x} = 20$

$e^{x} = \frac{20}{8}$

$e^{x} = \frac{5}{2}$

$\ln\left(\frac{5}{2}\right) = x$

$x = \ln\left(\frac{5}{2}\right)$

$x \doteq 0.92$

Laws of Logarithms apply to $y = \ln x$

a) ln 8 + ln 4

$= \ln 32$

b) 2 ln 5 – ln 6 + 3ln 3

$= \ln 25 - \ln 6 + \ln 27$

$= \ln\left(\frac{25}{6}\right) + \ln 27$

$= \ln\left(\frac{25(27)}{6}\right)$

$= \ln\left(\frac{225}{2}\right)$

$= \ln 225 - \ln 2$

Solving Functions

a) ln x – 5 = 0

$\ln x = 5$

$\boxed{e^{5} = x}$

b) $4 \ln x = \ln(x^2 + 12)$

$\ln x^4 = \ln(x^2+12)$

$x^2-4=0, \quad x^2+3 \neq 0$

$\boxed{x=2}, \; \cancel{x=-2}$

$\therefore \quad x^4 = x^2+12$

$x^4 - x^2 - 12 = 0$

$\ln(-2)$

$(x^2-4)(x^2+3)=0$

$e^? = -2$

c) $x \ln x (\ln x + 2) = 0$

$x \neq 0, \quad \ln x = 0, \quad \ln x + 2 = 0$

$e^0 = x \qquad \ln x = -2$

$\boxed{x=1} \qquad e^{-2} = x$

$\ln 0 \qquad e^? = 0 \qquad \boxed{x = \frac{1}{e^2}}$