MCV4U

Logarithmic Differentiation

We know that $\frac{d}{dx}e^x =$ e^x and $\frac{d}{dx}(\ln x) =$ $\frac{1}{x}$

The derivative for y = ln x enables us to find the derivatives of all other logarithmic functions. This process is called logarithmic differentiation.

Differentiation of Exponential Functions

Find the derivative of $y = a^x$

$\ln y = \ln a^x$

$\ln y = x \ln a$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = y \ln a$

$\boxed{dy/dx = a^x \ln a}$ *

Use this result to find the derivative of the following. Notice how you can either remember the result above, or simply take the "*ln of both sides*" to differentiate.

a) $y = 2^x$

$\frac{dy}{dx} = 2^x(\ln 2)$

or

$\ln y = \ln 2^x$

$\ln y = x \ln 2$

$\frac{1}{y}\frac{dy}{dx} = \ln 2$

$\frac{dy}{dx} = y \ln 2$

$dy/dx = 2^x \ln 2$

$y = a^x$

$\frac{dy}{dx} = a^x \ln a$

b) $y = 7^{x^2+5x}$

$\frac{dy}{dx} = 7^{x^2+5x}(\ln 7)(2x+5)$

$\ln y = \ln 7^{x^2+5x}$

$\ln y = (x^2+5x)\ln 7$

$\frac{1}{y}\frac{dy}{dx} = (2x+5)\ln 7$

$\frac{dy}{dx} = 7^{x^2+5x}(2x+5)(\ln 7)$

Logarithmic Differentiation can also be used to prove the power rule where *n is any real number.*
(We only proved it for positive integers)

$f(x) = x^3$

$f'(x) = 3x^2$

Prove that if n is any real number and $f(x) = x^n$, then $f'(x) = nx^{n-1}$

let $y = x^n$

$\ln y = \ln x^n$

$\ln y = n \ln x$

$\frac{1}{y}\frac{dy}{dx} = n\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \frac{yn}{x}$

$\frac{dy}{dx} = \frac{x^n n}{x^1}$

$$\boxed{\frac{dy}{dx} = nx^{n-1}}$$

Derivatives of Logarithmic Functions

$y = \log_a x$

$a^y = x$

$\ln a^y = \ln x$

$y \ln a = \ln x$

$y = \frac{\ln x}{\ln a}$

$\frac{dy}{dx} = \frac{1}{\ln a}\frac{1}{x}$

$$\boxed{\frac{dy}{dx} = \frac{1}{x \ln a}}$$

Use the result above to find the derivative of the following:

a) $y = \log_5 x$

$$\frac{dy}{dx} = \frac{1}{x\ln 5}$$

$\underline{\text{or}}$

$$5^y = x$$

$$y\ln 5 = \ln x$$

$$y = \frac{\ln x}{\ln 5}$$

$$y' = \frac{1}{x\ln 5} \leftarrow \frac{\frac{1}{x}}{\ln 5}$$

b) $y = \ln(x^2 + 3x)$

$$\frac{dy}{dx} = \frac{1}{x^2+3x}(2x+3)$$

$$\frac{dy}{dx} = \frac{(2x+3)}{x^2+3x}$$

c) $y = \log_6(x^2 + 3x)$

$$\frac{dy}{dx} = \frac{1}{(x^2+3x)\ln 6}(2x+3)$$

$$= \frac{(2x+3)}{(x^2+3x)\ln 6} \quad \#$$

$\underline{\text{or}}$

$$6^y = x^2 + 3x$$

$$y\ln 6 = \ln(x^2 + 3x)$$

$$y = \ln(x^2+3x)\frac{1}{\ln 6}$$

$$\frac{dy}{dx} = \frac{1}{(x^2+3x)\ln 6}(2x+3)$$

Sketch the following graph

$y = 2x(\ln x)^2$

$\ln 0$

$e^{?} = 0$

$\ln -2$

$e^{?} = -2$

$x > 0$

x-int $y = 0$

$0 = 2x(\ln x)^2$

$x \neq 0 \quad \ln x = 0$

$x = e^0$

$x = 1$

$(1, 0)$

As $x \to +\infty$

$y \to +\infty$

$y = 2x(\ln x)^2$

$\frac{dy}{dx} = 2(\ln x)^2 + 2\cancel{x}(2)(\ln x)\frac{1}{\cancel{x}}$

$\frac{dy}{dx} = 2(\ln x)^2 + 4\ln x$

$0 = 2\ln x(\ln x + 2)$

$\ln x = 0$

$\ln x = 0$

$x = 1$

$(1, 0)$

$\ln x + 2 = 0$

$\ln x = -2$

$x = e^{-2}$ or $x = \frac{1}{e^2}$

$y = \frac{2}{e^2}(\ln e^{-2})^2$

$y = \frac{2}{e^2}(-2)^2$

$y = \frac{8}{e^2}$

$\left(\frac{1}{e^2}, \frac{8}{e^2}\right) \simeq (0.14, 1.1)$

$\frac{dy}{dx} = 2(\ln x)^2 + 4\ln x$

$\frac{d^2y}{dx^2} = 4\ln x\left(\frac{1}{x}\right) + \frac{4}{x}$

$0 = \frac{4\ln x + 4}{x} \qquad x > 0$

$4\ln x + 4 = 0$

$\ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1}$ or $x = \frac{1}{e}$

$y = \frac{2}{e}$

inflection point at $\left(\frac{1}{e}, \frac{2}{e}\right) \simeq (.36, .74)$

turning points $(1, 0)$

$\left(\frac{1}{e^2}, \frac{8}{e^2}\right)$